Total Marks – 84
Attempt Questions 1-7
All questions are of equal value
Answer each question in a SEPARATE writing booklet.

Question 1 *(12 marks)* **Marks**

a) Solve for x: $\dfrac{1}{2x-1} \le 2$. **3**

b) Find $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$ **2**

c) Evaluate $\displaystyle\lim_{x\to 0} \frac{x}{\sin 3x}$. **2**

d) Use the substitution $u = 1+x$ to evaluate $\displaystyle\int_0^1 \frac{x}{\sqrt{1+x}}\,dx$. **3**

e) A curve has the parametric equations $x = \dfrac{t}{3}$, $y = 2t^2$. **2**

Find the Cartesian equation for this curve.

Question 2 *(12 marks)* Use a SEPARATE writing booklet.

a) (i) Differentiate $x\sin^{-1} x + \sqrt{1-x^2}$ **4**

(ii) Hence evaluate $\displaystyle\int_0^1 \sin^{-1} x\,dx$.

b) A particle is moving in simple harmonic motion. Its displacement x at time t is given by $x = 2\sin(3t)$, x in metres and t in seconds. **5**

(i) Find the period of the motion.
(ii) Find the maximum acceleration of the particle.
(iii) Find the speed of the particle when $x = 2$.

c) The volume, V of a spherical balloon of radius r mm is increasing at a constant rate of 100mm^3 per second. **3**

Given $V = \dfrac{4}{3}\pi r^3$ and $SA = 4\pi r^2$.

(i) Find $\dfrac{dr}{dt}$ in terms of r.

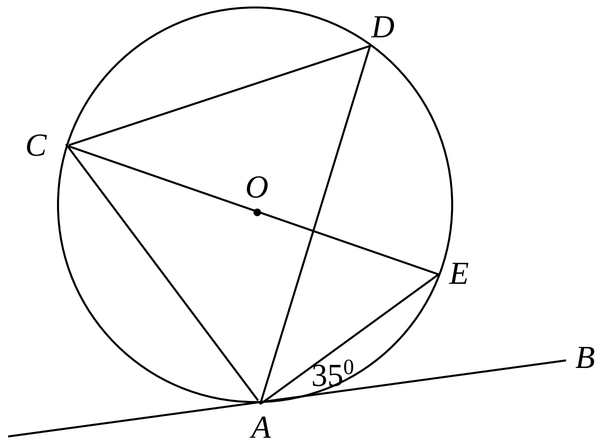
(ii) Determine the rate of increase of the surface area, S, of the sphere when the radius is 20mm.

Question 3 *(12 marks)* Use a SEPARATE writing booklet.

a) (i) Show that $f(x) = x - 3 + e^x$ has a root between $x = 0.7$ and $x = 0.9$. **1**

(ii) Starting with x = 0.8, use one application of Newton's method to find a better approximation for this root. Write your answer correct to three significant figures. **3**

b)



AB is a tangent and *CE* is a diameter to a circle, centre O. $\angle BAE = 35^\circ$ and D lies on the circumference as shown in the diagram.

(i) Find the size of $\angle ACE$, giving reasons. **1**

(ii) Find the size of $\angle ADC$. Justify your answer. **3**

c) Six backpackers arrive in a town with six hostels.

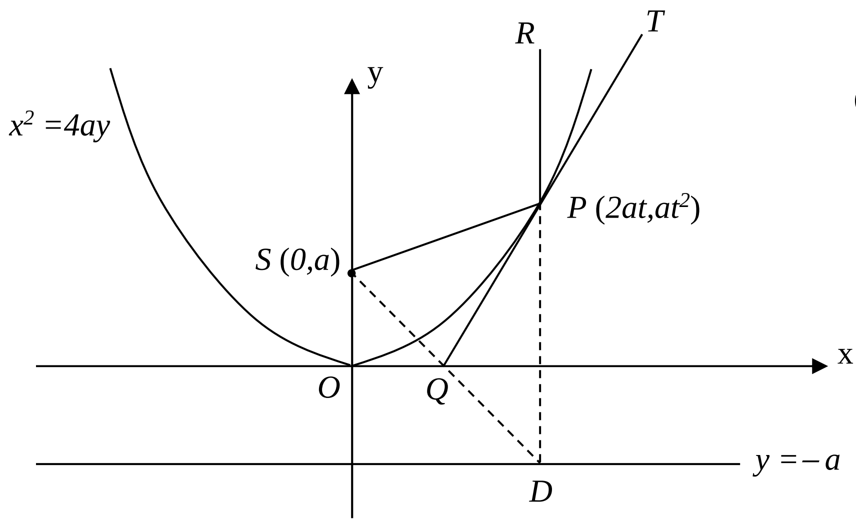
(i) How many different accommodation arrangements are there if there are no restrictions on which hostel each person can stay? **1**

(ii) How many different accommodation arrangements are there if each person stays in a different hostel? **1**

(iii) If three of the backpackers have been traveling together and must stay in the same hostel. How many different arrangements are there if the other three can go to any of the <u>other</u> hostels? **2**

Question 4 *(12 marks)* Use a SEPARATE writing booklet.

a) **6**



The diagram shows the parabola $x^2 = 4ay$ with focus $S\ (0,a)$ and directrix $y = -a$. The point $P\ (2at, at^2)$ is an arbitrary point on the parabola and the line RP is drawn parallel to the y axis, meeting the directrix at D. The tangent QPT to the parabola at P intersects SD at Q.

(i) Explain why $SP = PD$.

(ii) Find the gradient m_1 of the tangent at P.

(iii) Find the gradient m_2 of the line SD.

(iv) Prove that PQ is perpendicular to SD.

(v) Prove that $\angle RPT = \angle SPQ$.

b) Find the constant term in the expansion of $\left(x - \dfrac{1}{2x^3}\right)^{20}$. **3**

c) Show that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{8} - \frac{1}{4}$. **3**

Question 5 *(12 marks)* Use a SEPARATE writing booklet.

a) The velocity, v m/s, of a particle in Simple Harmonic Motion is given by $v^2 = 2x(6-x)$. **3**

(i) Find the acceleration of the particle in terms of x.

(ii) Prove that the particle always remains in the domain $0 \le x \le 6$.

b) A thermos container with a internal water temperature of $T\,^0$C loses heat when placed in a cooler environment, according to Newton's Law of Cooling, $\frac{dT}{dt} = k(T - T_0)$. Where t is the time elapsed in minutes and T_0 is the temperature of the environment in degrees Celsius. **5**

(i) The thermos with a internal water temperature of 95^0C is placed in an environment of -10^0C for 25minutes and cools to 85^0C. Find k.

(ii) How long would is take for the same thermos with initial temperature of 95^0C to lose 25% of its temperature when placed in an environment of 20^0C? (Assume k remains the same and give your answer correct to the nearest minute.)

c) The polynomial $2x^3 + ax^2 + bx + 1$ has $x+1$ as a factor and leaves a remainder of -5 when divided by $x-2$. Find the values a and b. **4**

Question 6 *(12 marks)* Use a SEPARATE writing booklet.

a) A biased coin has a probability of 0.6 of coming up heads. If this coin is tossed 7 times, find the probability of getting: (Give answers correct to 3 significant figures) **3**

(i) 7 heads.

(ii) exactly 4 tails.

b) Use mathematical induction to prove that, for $n \ge 1$, $3 + 7 + 11 + ... + (4n-1) = n(2n+1)$. **3**

c) The polynomial $P(x) = 2x^3 - 11x^2 + kx - 6$ has roots α, β, γ. **4**

(i) Find the value of $\alpha + \beta + \gamma$.

(ii) Find the value of $\alpha\beta\gamma$.

(iii) If the sum of two of the roots is 5, find the third root and hence find the value of k.

d) For the function $f(x) = \operatorname{cosec} x, 0 < x \le \frac{\pi}{2}$, state the domain and range of the inverse function $f^{-1}(x)$. **2**

Question 7 *(12 marks)* Use a SEPARATE writing booklet.

a) In a Rugby game, a player kicks a ball so that it travels in a parabolic path with initial angle of elevation of 80^0. He runs down the field and catches the ball 1 metre above the level at which the ball was initially projected. Five seconds elapses between the kick and the catch. **3**

Given: $x = Vt\cos 80, y = -5t^2 + Vt\sin 80$, calculate the horizontal distance the ball travels to the nearest metre.

b) (i) Show that $x^n(1+x)^n\left(1+\frac{1}{x}\right)^n = (1+x)^{2n}$. **1**

(ii) Hence prove that **3**

$$1+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n}.$$

c) On a large flat plane there is a communications tower and three huts, Hut A is due west of the tower, Hut C is due north of the tower and Hut B is on the line-of-sight from Hut A to Hut C. The angles of elevation to the top of the tower from Hut A, Hut B and Hut C are 52^0, 56^0 and 60^0 respectively. **5**

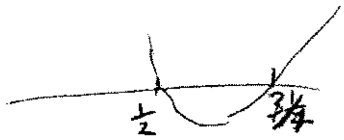
Determine the bearing of Hut B from the tower to the nearest degree.

End of paper

Question 1

a) $\frac{1}{2x-1} \times (2x-1)^2 \leq 2(2x-1)^2$ ✓

$2x-1 \leq 2(4x^2-4x+1)$



$0 \leq 8x^2-10x+3$

$0 \leq (2x-1)(4x-3)$ ✓

$\therefore \quad x < \frac{1}{2}$ or $x \geq 3/4$ ✓ ✓

b) $\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0$

$= \pi/4$ ✓

c) $\frac{1}{3}\lim_{x\to 0} \frac{3x}{\sin 3x} = \frac{1}{3}\times 1 = \frac{1}{3}$ ✓ ✓

d) $u = 1+x \qquad x = u-1$

$\frac{du}{dx} = 1 \qquad x=0 \quad u=1$

$du = dx \qquad x=1 \quad u=2$

$\int_0^1 \frac{x}{\sqrt{1+x}}\,dx = \int_1^2 \frac{u-1}{\sqrt{u}}\,dx$ ✓

$= \int_1^2 u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,dx$

$= \left[\frac{u^{3/2}}{3/2} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^2$ ✓

$= \left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^2$

$= \frac{2}{3}\times 2\sqrt{2} - 2\sqrt{2} - (2/3 - 2)$

$= \frac{4\sqrt{2}}{3} - 2\sqrt{2} + 1\tfrac{1}{3}$

$= \sqrt{2}\left(-\frac{2}{3}\right) + \frac{4}{3}$ ✓

$= \frac{4-2\sqrt{2}}{3}$

≈ 0.4

e) $x = \frac{t}{3}$ ①

$y = 2t^2$ ②

~~sub ① in ②~~

From ① $t = 3x$ ✓

sub $t = 3x$ in ②

$y = 2(3x)^2$

$y = 18x^2$ ✓

Ext 1 trial 2005

Q2

(a)

(i)

$$\frac{d}{dx}\left(x\sin^{-1}x+\left(1-x^2\right)^{\frac{1}{2}}\right)$$
$$=x.\frac{1}{\sqrt{1-x^2}}+\sin^{-1}x+\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}.-2x$$
$$=\sin^{-1}x$$

(ii)

$$\int_0^1 \sin^{-1}x\,dx$$
$$=\left[x\sin^{-1}x+\left(1-x^2\right)^{\frac{1}{2}}\right]_0^1$$
$$=\frac{\pi}{2}-1$$

(b)

(i) $\text{period}=\frac{2\pi}{3}$

(ii) $\dot{x}=6\cos 3t$

$\ddot{x}=-18\sin 3t$

Max acceleration is $18\text{m}/\text{s}^2$

(iii) when x=2 particle is at extreme value, and given SHM it is stationary.
Therefore $\dot{x}=0$

(c)

(i)

$$\frac{dV}{dr}=4\pi r^2$$
$$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$$
$$100=4\pi r^2\times\frac{dr}{dt}$$
$$\frac{dr}{dt}=\frac{25}{\pi r^2}\text{mm}/\text{s}$$

Ext 1 trial 2005

(ii)

$$\frac{dSA}{dr} = 8\pi r$$
$$\frac{dSA}{dt} = \frac{dSA}{dr} \times \frac{dr}{dt}$$
$$\frac{dSA}{dt} = 8\pi r \times \frac{25}{\pi r^2}$$
$$\frac{dSA}{dt} = \frac{200}{r}$$

when $r = 20$

$$\frac{dSA}{dt} = 10\text{mm}^2/\text{s}$$

Ext 1 Solutions '05 Q3.

(3) (a)(i) $f(0.7) < 0$
$f(0.9) > 0$] (1) both

(ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $\quad f(x) = x - 3 + e^x$ $\quad f'(x) = 1 + e^x$ (1) sub in correct expression.

$= 0.8 - \frac{0.8 - 3 + e^{0.8}}{1 + e^{0.8}}$ (1) $f'(0.8)$

$= 0.792$ (1) Correct answer

(b)(i) $A\hat{C}E = 35°$ (alternate segment theorem) (1) reason

(ii) $C\hat{A}E = 90°$ ($\angle$ in semi-circle) (1) reason

$C\hat{E}A = 180° - (90° + 35°)$
$= 55°$ (1) 90° and 55°

$A\hat{D}C = C\hat{E}A$ ($\angle$s in same segment)
$= 55°$ (1) Ans & reason.

(c) (i) 6^6 (1)

(ii) $6!$ (1)

(iii) $6 \times 5^3 = 750$ (1) $6 \times$ (some quantity) (1) 5^3

Q4

a) i/ The parabola is defined as the set of points equidistant from a point (the focus, S) and ~~the~~ a line (the directrix). Therefore, if it is a parabola, then $SP = PD$. (1)

1 - Uses definition of a parabola

ii/ $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

when $x = 2at$,

$m = \frac{4at}{4a}$ (1)

$\boxed{m_1 = t.}$

1 - correct differentiation and substitution.

iii/ $S(0,a)$ $D(2at,-a)$

$m = \frac{-a-a}{2at-0}$ (1)

$= \frac{-2a}{2at}$

$= -\frac{1}{t}$

1 - correct point & correct use of gradient formula.

iv/ $m_{PQ} = t$ $m_{SD} = -\frac{1}{t}$

$\therefore m_{PQ} \cdot m_{SD} = t \cdot -\frac{1}{t}$

$= -1$ (1)

$\therefore PQ \perp SD$

1 - correct reasoning

v/ In ΔSPQ and ΔDPQ,

$\angle PQS = \angle PQD = 90°$ (adj supp)

$SP = PD$ (from i) (1)

$\therefore \angle PSQ = \angle PDQ$ (ΔSPD isosc.)

$\therefore \angle SPQ = \angle DPQ$ ($\angle$ sum Δs)

Now $\angle DPQ = \angle RPT$ (vert opp) (1)

$\therefore \angle SPQ = \angle RPT$

1 - Shows $\angle SPQ = \angle DPQ$ or $\angle DPQ = \angle RPT$

2 - Shows both.

b) General term of $\left(x - \frac{1}{2x^3}\right)^{20}$

$= {}^{20}C_r \; x^r \left(-\frac{1}{2x^3}\right)^{20-r}$

$= {}^{20}C_r \; x^r . \left(\frac{-1}{2}\right)^{20-r} x^{-3(20-r)}$

$= {}^{20}C_r \; x^{-60+4r} . \left(-\frac{1}{2}\right)^{20-r}$ ①

1 - finds general term

For constant term

$-60 + 4r = 0$

$4r = 60$

$r = 15$ ①

1 - finds relevant r.

$\therefore$ Constant term $= {}^{20}C_{15} \left(-\frac{1}{2}\right)^5$

$= 15504 . -\frac{1}{32}$ ①

$= 484.5$.

1 - evaluates constant term (Guess max 2 marks)

c) $\int_{\pi/4}^{\pi/2} \cos^2 x \, dx = \frac{1}{2} \int_{\pi/4}^{\pi/2} \cos 2x + 1 \, dx$ ①

1 - substitute identity

$= \frac{1}{2} \left[\frac{1}{2} \sin 2x + x \right]_{\pi/4}^{\pi/2}$ ①

1 - correct integrand

$= \frac{1}{2} \left(0 + \pi/2 - \left(\frac{1}{2} + \frac{\pi}{4} \right) \right)$

①

1 - evaluation.

$= \pi/4 - \frac{1}{4} - \frac{\pi}{8}$

$= \frac{\pi}{8} - \frac{1}{4}$

$\therefore \int_{\pi/4}^{\pi/2} \cos^2 x \, dx = \frac{\pi}{8} - \frac{1}{4}$ //.

Ext 1 trial 2005

Q5

(a)

(i)

$$\frac{1}{2}v^2 = 6x - x^2$$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6 - 2x$$

(ii)

since $v^2 \geq 0, \quad 2x(x-6) \geq 0$

Therefore $0 \leq x \leq 6$

(b) (i) If $\frac{dT}{dt} = k(T - T_0)$ then $T = T_0 + Ae^{kt}$

Since ambient temp is -10, $T_0 = -10$

When t=0, T=95

Therefore 95=-10+A

A=105

i.e. $T = -10 + 105e^{kt}$

when $t = 25, \quad T = 85$

$$85 = -10 + 105e^{25k}$$

$$e^{25k} = \frac{95}{100}$$

$$k = \frac{\ln\left(\frac{19}{21}\right)}{25}$$

(ii) If ambient temp is 20, $T = 20 + Be^{kt}$

when t=0, T=95, Therefore B=75

i.e. $T = 20 + 75e^{kt}$

when $T = 0.75 \times 95$

$$71.25 = 20 + 75e^{kt}$$

$$e^{kt} = \frac{51.25}{75}$$

$$t = \frac{\ln\left(\frac{51.25}{75}\right)}{k}$$

$t = 95\text{mins}$ (nearest min)

Ext 1 trial 2005

(c)

$P(x) = 2x^3 + ax^2 + bx + 1$

$P(-1) = -2 + a - b + 1 = 0$

i.e. $a - b = 1$ $\boxed{1}$

$P(2) = 16 + 4a + 2b + 1 = -5$

i.e. $2a + b = -11$ $\boxed{2}$

Solving $\boxed{1}$ and $\boxed{2}$ simultaneously

$3a = -10$

$a = -\frac{10}{3}, \quad b = -\frac{13}{3}$

Ext 1 trial 2005

Q6

(a) (i)

$P(7\text{ heads}) = (0.6)^7$

$= 0.280 \quad (3s.f)$

(ii) $P(4\text{ tails}) = {}^7C_4(0.4)^4(0.6)^3$

$= 0.194 \quad (3s.f)$

(b) Show true for n=1

LHS=3 $\quad$ $RHS = 1(2\times1+1)$

$= 3$

If true for n=k, then

$3+7+\dots\dots\dots+(4k-1) = k(2k+1) \quad \otimes$

Show true for n=k+1

i.e. $3+7+\dots\dots\dots+(4k-1)+4(k+1)-1 = (k+1)(2(k+1)+1)$

$LHS = 3+7+\dots\dots\dots+(4k-1)+(4k+3)$

$= k(2k+1)+(4k+3) \quad \text{using} \otimes$

$= 2k^2+k+4k+3$

$= (2k+3)(k+1)$

$= RHS$

Since it is true for n=1 and, if true for n=k, it is true for n=k+1, then by the Principle of Mathematical Induction it is true for all integer $n \geq 1$.

(c) (i) $\alpha+\beta+\gamma = \frac{11}{2}$

(ii) $\alpha\beta\gamma = 3$

(iii) If $\alpha+\beta = 5, \quad \gamma = \frac{11}{2} - 5$ from (i)

i.e. $\gamma = \frac{1}{2}$

Also $\alpha\beta+\alpha\gamma+\beta\gamma = \frac{k}{2}$

Now $\alpha\beta\times\frac{1}{2} = 3$ i.e. $\alpha\beta = 6$

and $\gamma(\alpha+\beta) = \frac{1}{2}\times5$

$\therefore \quad \alpha\beta+\alpha\gamma+\beta\gamma = \frac{k}{2} = 6+\frac{1}{2}\times5$

i.e. $\quad k = 17$

(d) domain $x \geq 1$

range $0 < y < \frac{\pi}{2}$

c) $P(x) = 2x^3 + ax^2 + bx + 1$

$P(-1) = -2 + a - b + 1 = 0$ ✓

$a - b = 1$ ①

$P(2) = 16 + 4a + 2b + 1 = -5$ ✓

$2a + b = -11$ ②

Solving ① & ② simultaneously. ✓

$3a = -10$

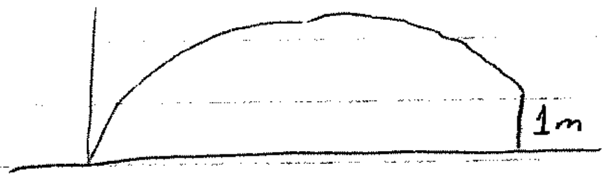
$a = \frac{-10}{3}$ ✓

$b = \frac{-13}{3}$ (4)

Trial HSC 2005

Q7

a)



$y = -5t^2 + Vt\sin 80$

$t = 5 \quad y = 1$

$1 = -125 + 5V\sin 80$

$V = \frac{126}{5\sin 80}$ ✓

$x = Vt\cos 80$

$= \frac{126}{5\sin 80} \times 5\cos 80$ ✓

$\approx 22\text{ m}$ ✓

b) i) $LHS = x^n(1+x)^n\left(1+\frac{1}{x}\right)^n$

$= (1+x)^n\left(x\left(1+\frac{1}{x}\right)\right)^n$ ✓

$= (1+x)^n(x+1)^n$

$= (1+x)^{2n}$

ii) By considering the coefficent of x^n

for $(1+x)^{2n}$ coefficent of $x^n = {}^{2n}C_n$ ✓

for $x^n(1+x)^n\left(1+\frac{1}{x}\right)^n$ coefficent of x^n is the constant terms of $(1+x)^n\left(1+\frac{1}{x}\right)^n$ ✓

$= {}^nC_0 \times {}^nC_0 + {}^nC_1 \times {}^nC_1 + \ldots + {}^nC_n \times {}^nC_n$

$= ({}^nC_0)^2 + ({}^nC_1)^2 + \ldots + ({}^nC_n)^2$ ✓

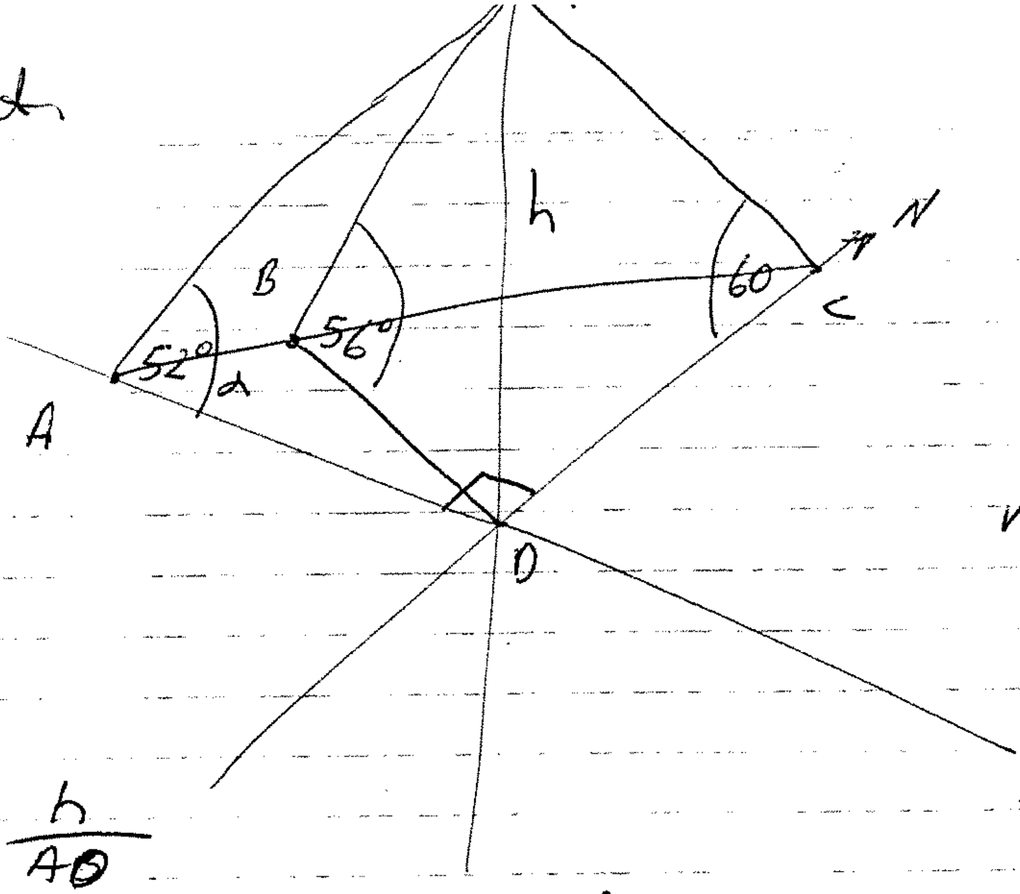
$= 1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2$

By equating coefficent of x^n

$1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$

Question 7 contd

c)



In Δ ADT

$$\tan 52^\circ = \frac{h}{AD}$$

$$AD = \frac{h}{\tan 52}$$

In Δ BDT

$$\tan 56 = \frac{h}{BD}$$

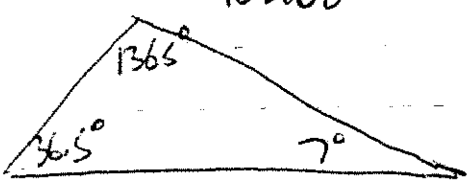
$$BD = \frac{h}{\tan 56}$$

✓

In Δ CDT

$$\tan 60 = \frac{h}{DC}$$

$$DC = \frac{h}{\tan 60}$$



$\therefore$ Bearing is 277°

In Δ ADC

$$\tan \angle DAC = \frac{DC}{AD}$$

$$= \frac{\frac{h}{\tan 60}}{\frac{h}{\tan 52}}$$

$$= \frac{\tan 52}{\tan 60}$$ ✓

$$\angle ADC = 36.463^\circ$$

$$\frac{\sin \angle ABD}{AD} = \frac{\sin \angle BAD}{BD}$$

$$\sin \angle ABD = \frac{AD \sin\left(\frac{\tan 52}{\tan 60}\right)}{BD}$$

$$= \frac{\frac{h}{\tan 52} \sin\left(\frac{\tan 52}{\tan 60}\right)}{\frac{h}{\tan 56}}$$

$$= \frac{\tan 56}{\tan 52} \sin\left(\frac{\tan 52}{\tan 60}\right)$$

$\angle ABD = 136.5^\circ$ (acute option does not work)